EXISTENCE OF BOUND STATES FOR LAYERS BUILT OVER HYPERSURFACES IN $\mathbb{R}^{n+1}$

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CONTENTS

1. INTRODUCTION

In their study of the spectrum of quantum layers [6], Duclos, Exner, and Krejčiřík proved the existence of bound states for certain quantum layers [1]. Part of their motivations to study the quantum layers is from mesoscopic physics. From the mathematical point of view, a quantum layer is a noncompact noncomplete manifold. For such a manifold, the spectrum of the Laplacian (with Dirichlet or Neumann boundary condition) is less understood. Nevertheless, from [6] and this paper, we shall see that the spectrum of a quantum layer has very interesting properties not only from the point of view of physics but also from the point of view of mathematics.

Mathematicians are interested in the spectrum of two kinds of manifolds: compact manifolds (with or without boundary), and noncompact complete manifolds. For these two kinds of manifolds, one can prove [13, 14] that the Laplacians can be uniquely extended as self-adjoint operators from operators on smooth functions with compact support. For a compact manifold, by the Hodge theorem, we can prove that the spectrum of the Laplacian is composed of only discrete spectrum. On the other hand, the spectrum of Laplacian of a noncompact complete manifold is rather complicated. In general it has both discrete and essential spectrum.

In general, it is rather difficult to prove the existence of discrete spectrum for a noncompact manifold, because the existence of an L^2 eigenfunction is a highly nontrivial fact. However, in the following special case, the discrete spectrum does exist.

We define the following two quantities:

[1]Quantum layers were studied by many authors. An incomplete list of the works are [1, 2, 5, 7, 8, 9, 10, 11, 12, 21, 23, 22].

Definition 1.1. *Let M be a manifold whose Laplacian Δ can be extended to a self-adjoint operator. Let*

$$\sigma_0 = \inf_{f \in C_0^\infty(M)} \frac{-\int_M f \Delta f}{\int_M f^2}, \tag{1.1}$$

$$\sigma_{\text{ess}} = \sup_K \inf_{f \in C_0^\infty(M \setminus K)} \frac{-\int_M f \Delta f}{\int_M f^2}, \tag{1.2}$$

where K is running over all compact subsets of M.

σ_0 is the lower bound of the spectrum and σ_{ess} is the lower bound of the essential spectrum. In general, we have $\sigma_0 \leq \sigma_{\text{ess}}$. If $\sigma_0 < \sigma_{\text{ess}}$, then the set of discrete spectrum must be nonempty. In particular, since the spectrum of a self-adjoint operator is a closed subset of the real line, there is an L^2 smooth function f of M such that

$$\Delta f = -\sigma_0 f.$$

(σ_0, f) is called the ground state of the Laplacian.

We don't expect $\sigma_0 < \sigma_{\text{ess}}$ to be true in general. It seems that more often we would get the opposite result $\sigma_0 = \sigma_{\text{ess}}$. For example, by a theorem of Li and Wang [27, Theorem 1.4], we know that if the volume growth of a complete manifold is sub-exponential and if the volume is infinite, then $\sigma_{\text{ess}} = 0$. Thus in that case, $\sigma_0 = \sigma_{\text{ess}} = 0$ and we don't know a general way to prove the existence of ground state.

Let Σ be an oriented n-manifold and $\pi : \Sigma \to \mathbb{R}^{n+1}$ be an immersion. Let N be the unit normal vector field of Σ, we can define the following map

$$p : \Sigma \times (-a, a) \to \mathbb{R}^{n+1}, \quad (x, u) \mapsto x + uN,$$

where a is a small positive number such that the map p is an immersion.

The quantum layer Ω built over Σ, as a differentiable manifold, is very simple: $\Omega = \Sigma \times (-a, a)$. The Riemannian metric on Ω is defined by $p^*(ds_E^2)$, where ds_E^2 is the Euclidean metric of $\mathbb{R}^{n+1}$. The number a is called the depth of the layer.

Remark 1.1. The setting above is a little bit more general than in the paper [6], where the authors require that both Σ and Ω are embedded. In particular, they assume that the quantum layers are not self-intersecting. There are some advantages of our treatment: first, all the theorems still remain true in immersed cases, and second, it is possible to estimate the range of the depth a using the upper bound of the second fundamental form in the case of immersion, while in the embedded case, global conditions of (Σ, π) must be imposed in order to keep the layers from self-intersecting.

The aim of this paper is to study the ground state of the noncompact noncomplete Riemannian manifold $(\Omega, p^*(ds_E^2))$, where we assume the Dirichlet boundary condition for the Laplacian. Our work is clearly motivated by the work of [6].

The first main result of this paper is the existence of the ground state of layer over convex surface in $\mathbb{R}^3$. We are motivated by the following result in [6]:

Theorem (Duclos, Exner, and Krejčiřík)**.** *Let Ω be a layer of depth a over a surface of revolution whose Gauss curvature is integrable. Suppose Ω is not self-intersecting, and suppose $a\|A\| < C_0 < 1$, where $\|A\|$ is the norm of the second fundamental form and C_0 is a constant. If the surface is not totally geodesic, then $\sigma_0 < \pi^2/(4a^2)$.*

Overlapping with the above result, we proved the following

Theorem 1.1. *Let Σ be a convex surface in $\mathbb{R}^3$ which can be represented by the graph of a convex function $z = f(x, y)$. Suppose 0 is the minimum point of the function and suppose that*

at 0, f is strictly convex. Furthermore suppose that the second fundamental form goes to zero at infinity. Let C be the supremum of the second fundamental form of Σ. Let $Ca < 1$. Then the ground state of the quantum layer Ω exists.

Remark 1.2. We let Σ to be the surface defined by the function
$$f(x, y) = x^2 + y^2.$$
A straightforward computation gives the mean curvature of Σ:
$$H = 4 \cdot \frac{1 - \frac{2(x^2 + y^2)}{1 + 4(x^2 + y^2)}}{\sqrt{1 + 4(x^2 + y^2)}}.$$
Thus H and then the second fundamental form goes to zero at infinity. By the above theorem, the quantum layer built from the above surface has a ground state.

The second main result of this paper is motivated by the following:

Theorem (Duclos, Exner, and Krejčiřík). *Let Σ be a C^2-smooth complete simply connected noncompact surface with a pole embedded in $\mathbb{R}^3$. Let the layer Ω built over the surface be not self-intersecting. If the surface is not a plane but it is asymptotically planar, and if the Gauss curvature is integrable and the total Gauss curvature is nonpositive, then the ground state exists.*

In a more recent paper [2], Carron, Exner, and Krejčiřík observed that the assumptions of simply-connectedness and the existence of a pole on Σ can be removed. Σ is allowed to have a rather complicated topology.

By a theorem of Huber [20], Σ is conformally equivalent to a compact Riemann surface with finitely many points removed. In particular, we have
$$\int_\Sigma K \leq 2\pi e(\Sigma),$$
where $e(\Sigma)$ is the Euler characteristic number of Σ. The deficit of the above inequality can be represented by isoperimetric constants. Let $E_1, \cdots, E_s$ be the ends of the surface Σ. For each E_i, we define

(1.3)
$$\lambda_i = \lim_{r \to \infty} \frac{A_i(r)}{\pi r^2},$$

where $A_i(r)$ is the area of the ball $B(r) \cap E_i$. We have the following

Theorem (Hartman [19]). *Using the above notations, we have*
$$\frac{1}{2\pi} \int_\Sigma K = e(\Sigma) - \sum \lambda_i.$$

$\square$

We have the following [2]

Theorem 1.2. *Suppose that Σ is a complete immersed surface of $\mathbb{R}^3$ such that the second fundamental form $A \to 0$. Suppose that the Gauss curvature is integrable and suppose that*

(1.4)
$$e(\Sigma) - \sum \lambda_i \leq 0,$$

where λ_i is the isoperimetric constant at each end defined in (1.3). Let a be a positive number such that $a\|A\| < C_0 < 1$. If Σ is not totally geodesic, then the ground state of the quantum layer Ω exists. In particular, if $e(\Sigma) \leq 0$, then the ground state exists.

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[2]There is an overlap of this result with the one in [2]. The proofs are similar but not identical. In particular, we use the result of Hartman instead of the Kohn-Vossen inequality.

More generally, one of the key observation of our paper is that in order to generalize the results in [6] to high dimensions, we must assume the parabolicity of the hypersurface Σ. The parabolicity of complete manifold was introduced by Li and Tam [26] (see also the survey papers [24, 25]). A surface with a pole and L^1 Gauss curvature is parabolic. Thus the following result is a high dimensional generalization of the above result of Duclos, Exner, and Krejčiřík:

Theorem 1.3 (**Main theorem**). *Let $n \geq 2$ be a natural number. Suppose $\Sigma \subset \mathbb{R}^{n+1}$ is a complete immersed parabolic hypersurface such that the second fundamental form $A \to 0$ at infinity. Moreover, we assume that*

$$(1.5) \qquad \sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) \text{ is integrable and } \quad \int_{\Sigma} \sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) d\Sigma \leq 0,$$

where $\mu_{2k} > 0$ for $k \geq 1$ are coefficients defined in Lemma 5.1, $[n/2]$ is the integer part of $n/2$, and $\mathrm{Tr}(\mathcal{R}^k)$ is defined in (5.22). Let a be a positive real number such that $a\|A\| < C_0 < 1$ for a constant C_0. If Σ is not totally geodesic, then the ground state of the quantum layer Ω exists.

Corollary 1.1. *Let ρ be the scalar curvature of Σ. If $n = 3$, then the main conditions (1.5) in Theorem 1.3 become*

(1) ρ *is integrable;*
(2) $\int_{\Sigma} \rho \, d\Sigma \leq 0$,

If $n = 4$, and if the sectional curvature of Σ is positive outside a compact set of Σ, then the conditions (1.5) become

(1) ρ *is integrable;*
(2) $\int_{\Sigma} \rho \, d\Sigma + 16(\frac{\pi^2}{6} - 1)a^3 e(\Sigma) \leq 0$,

where $e(\Sigma)$ is the Euler characteristic number of Σ.

The organization of the paper is as follows: in §2, we define the quantum layers and give their basic properties; in §3, we give the lower bound of the essential spectrum of a quantum layer; in §4, the parabolicity of a submanifold of $\mathbb{R}^{n+1}$ is introduced; in §5, Theorem 1.2, The main theorem (Theorem 1.3), and Corollary 1.1 are proved; finally, in §6, Theorem 1.1 is proved.

We end up this section by posing the following question:

Let Σ be a noncompact complete Riemannian manifold of dimension n. Then what do we have to assume on Σ so that when $\Sigma \to \mathbb{R}^{n+1}$ is an asymptotically flat but not totally geodesic immersion, the layer Ω built over Σ has ground state?

In particular, if $n = 2$, the works of [6, 2] suggest that the quantum layer Ω should have ground state when the Gauss curvature is integrable[3].

2. Geometry of quantum layers

Let $n > 1$ be an integer and let Σ be an immersed (oriented) hypersurface of $\mathbb{R}^{n+1}$. Let $a > 0$ be a real number. Heuristically speaking, the quantum layer Ω is obtained by fattening Σ by a thickness of a in the directions of N and $-N$, respectively, where N is the unit normal vector

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[3]This part of the question was implied in [6]. By [6] and this paper, we just need to show that for layers bulit over simply-connected surfaces with positive total Gauss curvature, the ground state exists.

field. As a differentiable manifold, Ω is just $\Sigma \times (-a, a)$. We impose the following assumptions on Σ and Ω:

A1). Let A be the second fundamental form of Σ. We regard A as a linear operators on $T_x \Sigma$ for every $x \in \Sigma$. We assume that there is a constant C_0 such that $a||A||(x) < C_0 < 1$.

A2). $||A||(x) \to 0$ as $d(x, x_0) \to \infty$, where $x_0 \in \Sigma$ is a fixed point.

Definition 2.1. *Let $x_1, \cdots, x_n$ be a local coordinate system of Σ. Then $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial u})$ is a local frame of Ω, where $u \in (-a, a)$. Such a local coordinate system of Ω is referred as a standard coordinate system of Ω in this paper.*

We consider the map

$$p : \Sigma \times (-a, a) \to \mathbb{R}^{n+1}, \quad (x, u) \mapsto x + uN.$$

Let $y_1, \cdots, y_{n+1}$ be the Euclidean coordinates of $\mathbb{R}^{n+1}$. Let

$$ds_E^2 = dy_1^2 + \cdots + dy_{n+1}^2$$

be the Euclidean metric of $\mathbb{R}^{n+1}$. Let G_{ij} $(i, j = 1, \cdots, n+1)$ be defined by

$$\sum_{i,j=1}^n G_{ij} dx_i dx_j + \sum_{i=1}^n G_{i,n+1} dx_i du + \sum_{i=1}^n G_{n+1,i} du dx_i + G_{n+1,n+1} du du = p^*(ds_E^2).$$

If p is nonsingular at a point, then the matrix G_{ij} is positive definite at that point. In order to express G_{ij} in term of the geometry of Σ, we introduce the following notations:

Let (h_{ij}) $(i, j = 1, \cdots n)$ be the matrix representation of the second fundamental form A with respect to the local frame $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$. Let $g_{ij} dx_i dx_j = p^*(ds_E^2)$ be the Riemannian metric of Σ. Let $h_j^\sigma = g^{\sigma i} h_{ij}$. Then a straightforward computation gives (cf. [6]):

$$(2.1) \qquad G_{ij} = \begin{cases} (\delta_i^\sigma - u h_i^\sigma)(\delta_\sigma^\rho - u h_\sigma^\rho) g_{\rho j} & 1 \le i, j \le n \\ 0 & i \text{ or } j = n+1 \\ 1 & i = j = n+1 \end{cases}.$$

In particular, we have

$$(2.2) \qquad \det(G_{ij}) = (\det(1 - uA))^2 \det(g_{ij}),$$

and

$$(2.3) \qquad \begin{aligned} \det(I - uA) &= \prod_{i=1}^n (1 - u\lambda_i) \\ &= 1 - u \sum_{i=1}^n \lambda_i + u^2 \sum_{i<j} \lambda_i \lambda_j - u^3 \sum_{i<j<l} \lambda_i \lambda_j \lambda_l + \cdots + (-1)^n u^n \prod_{i=1}^n \lambda_i, \end{aligned}$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues, or the principal curvatures of the second fundamental form A. In a more intrinsic way, let $c_k(A)$ be the k-th elementary polynomial of A. Then we have

$$(2.4) \qquad \det(I - uA) = \sum_{i=0}^n (-1)^k u^k c_k(A),$$

where we define $c_0(A) = 1$.

The following lemma is elementary but important:

Lemma 2.1. *Using the above notations and under Assumption A1), we have*

$$(1 - |u| \cdot ||A||)^n \le |\det(I - uA)| \le (1 + |u| \cdot ||A||)^n.$$

The proof is elementary and is omitted.

$\square$

Corollary 2.1. *We adopt the above notations and Assumption A1). Then the map p is an immersion. In that case, $p^*(ds_E^2)$ is a Riemannian metric on Ω. Let $d\Omega$ be the measure defined by the metric and let $dud\Sigma$ be the product measure on Ω. Then we have*

$$(2.5) \qquad (1 - |u|||A||)^n dud\Sigma \le d\Omega \le (1 + |u|||A||)^n dud\Sigma.$$

Proof. By Assumption A1), (2.2) and Lemma 2.1, we know that $\det(G_{ij}) > 0$. Thus p is nonsingular. (2.5) follows from Lemma 2.1 directly.

$\square$

Definition 2.2. *We define the quantum layer to be the Riemannian manifold $(\Omega, p^*(ds_E^2))$, where ds_E^2 is the standard Euclidean metric of $\mathbb{R}^{n+1}$. The real numbers a and $d = 2a$ are called the depth and the width of the quantum layer, respectively.*

The Laplacian $\Delta = \Delta_\Omega$ can be written as

$$(2.6) \qquad \Delta = \frac{1}{\sqrt{\det(G_{kl})}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(G^{ij} \sqrt{\det(G_{kl})} \frac{\partial}{\partial x_j} \right) + \frac{1}{\sqrt{\det(G_{kl})}} \frac{\partial}{\partial u} \left(\sqrt{\det(G_{kl})} \frac{\partial}{\partial u} \right),$$

where $(x_1, \cdots, x_n, u)$ is the local coordinates defined in Definition 2.1. We have

$$(2.7) \qquad (\Delta F, G) = (F, \Delta G) \quad \forall F, G \in C_0^\infty(\Omega),$$

where $(\cdot, \cdot)$ is the L^2 inner product

$$(2.8) \qquad (F, G) = \int_\Omega FG d\Omega.$$

The norm $||F||$ is defined as $\sqrt{(F, F)}$. If F, G are differentiable, we define

$$(2.9) \qquad (\nabla F, \nabla G) = \int_\Omega \left(\sum_{i,j=1}^n G^{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} + \frac{\partial F}{\partial u} \frac{\partial G}{\partial u} \right) d\Omega.$$

Also, we define $||\nabla F|| = \sqrt{(\nabla F, \nabla F)}$.

In the case of compact manifold or noncompact complete manifold, the self-adjointness of the Laplacians is classical [13, 14]. A quantum layer is a noncompact noncomplete manifold. For such a manifold, we still have

Proposition 2.1. Δ *can be extended as a self-adjoint operator.*

Proof. According to [28], we define the Hilbert space H_1 to be the closure of the space $C_0^\infty(\Omega)$ under the norm

$$||F||_{H_1} = \sqrt{||F||^2 + ||\nabla F||^2}.$$

We define the sesquilinear form

$$Q(F, G) = (\nabla F, \nabla G),$$

for functions $F, G \in H_1$. By [28, Theorem VIII.15], Q is the quadratic form of a unique self-adjoint operator. Such an operator is an extension of Δ, which we still denote as Δ. Furthermore, by the relation of Δ with the quadratic form, we can verify that σ_0 and σ_{ess} in (1.1), (1.2) are the infimum of the spectrum and the essential spectrum of Δ, respectively.

$\square$

3. LOWER BOUND OF THE ESSENTIAL SPECTRUM.

The boundaries of Ω are $\Sigma \times \{\pm a\}$, which are smooth manifolds. It is not hard to see that (1.2) can be written as

$$\sigma_{\text{ess}} = \lim_{i \to \infty} \inf\{\frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2} \mid f \in C_0^\infty(\Omega \setminus K_i)\},$$

where $\{x_0\} \subset K_1 \subset K_2 \subset \cdots$ is any compact exhaustion of Ω. For example, we can take

$$K_i = \{x + uN \mid x \in \overline{B_{x_o}(i)} \subset \Sigma, u \in \left[\frac{-a(i-1)}{i}, \frac{a(i-1)}{i}\right]\}.$$

To perform our estimate we need to obtain a lower bound for the Rayleigh quotient $\frac{\int |\nabla f|^2}{\int f^2}, \forall f \in C_0^\infty(\Omega \setminus K_i)$ for a large enough $i \in \mathbb{N}$.

We use the standard coordinate system $(x_1, \cdots, x_n, u)$ of Definition 2.1. Let $f_j = \frac{\partial f}{\partial x_j}$, $(i = 1, \cdots n)$ and $f_{n+1} = \frac{\partial f}{\partial u}$. Then

$$|\nabla f|^2 = |f_{n+1}|^2 + \sum_{k,l \neq n+1} G^{kl} f_k f_l,$$

where G^{ij} is the inverse of G_{ij}. In particular, we have

(3.1) $$|\nabla f|^2 \geq \left|\frac{\partial f}{\partial u}\right|^2.$$

Since $f = 0$ on $\partial\Omega$, the Poincaré inequality gives

(3.2) $$\int_{-a}^a \left|\frac{\partial f}{\partial u}\right|^2 du \geq \kappa_1^2 \int_{-a}^a f(u)^2 du,$$

where $\kappa_1 = \pi/2a$.

Theorem 3.1. *Under Assumptions A1) and A2), we have $\sigma_{\text{ess}} \geq \kappa_1^2$.*

Proof. We first observe that for arbitrary $\varepsilon > 0$, there is an i large enough such that $||A|| < \varepsilon$ on $\Sigma \setminus K_i$. By Corollary 2.1, we know that

(3.3) $$(1 - a\varepsilon)^n du d\Sigma \leq d\Omega \leq (1 + a\varepsilon)^n du d\Sigma.$$

Thus we have

(3.4) $$\int_\Omega f^2 d\Omega \leq (1 + a\varepsilon)^n \int_\Sigma \int_{-a}^a f^2 du d\Sigma.$$

On the other hand, by (3.1), (3.2) and (3.3), we have

(3.5) $$\int_\Omega |\nabla f|^2 d\Omega \geq (1 - a\varepsilon)^n \kappa_1^2 \int_\Sigma \int_{-a}^a f^2 du d\Sigma.$$

Comparing (3.4) and (3.5), we have

$$\sigma_{\text{ess}} \geq \frac{(1 - a\varepsilon)^n}{(1 + a\varepsilon)^n} \kappa_1^2.$$

Since ε is arbitrary, we get the conclusion of the theorem.

$\square$

4. Parabolicity of complete Riemannian manifolds

Before giving the formal definition, we study the following example.

Suppose $n > 1$ is an integer. Let $R > 0$ be a big number. We are interested in the set of functions

$$F(R) = \{f \in C_0^n(\mathbb{R}^n) | f \equiv 1 \text{ for } |x| < R, f \text{ is rotationally symmetric.}\}.$$

We have the following

Example 1. *If $n > 2$, then for any $C > 0$ there exists an R_0 such that for any $R > R_0$ we have*

$$\int_{\mathbb{R}^n} |\nabla f|^2 > C$$

for all $f \in F(R)$. If $n = 2$, then for any $\varepsilon > 0$ there exists $R_0 > 0$ such that for any $R > R_0$, we can find an $f_R \in F(R)$ for which

$$\int_{\mathbb{R}^2} |\nabla f|^2 < \varepsilon.$$

Proof. If $n > 2$, then

$$\int_R^\infty 1/r^{n-1} dr = \frac{1}{n-2} \cdot \frac{1}{R^{n-2}}.$$

Thus we have

$$\int_{\mathbb{R}^n} |\nabla f|^2 \geq (n-2)cR^{n-2} \int_R^\infty r^{n-1} \left|\frac{\partial f}{\partial r}\right|^2 dr \int_R^\infty 1/r^{n-1} dr \geq (n-2)cR^{n-2} \to +\infty$$

by Cauchy inequality, where c is the volume of the unit $(n-1)$-sphere. However, for $n = 2$, we let $f_R = \sigma_R(|x|) \in F(R)$, where $\sigma_R(t)$ is defined as

$$\sigma_R(t) = \begin{cases} 1 & t \leq R \\ (1 - \frac{\log R}{R})^{-1}(\frac{\log R}{\log t} - \frac{\log R}{R}) & R < t \leq e^R \\ 0 & t \geq e^R \end{cases}.$$

A straightforward computation gives

$$\int_0^\infty t|\sigma_R'(t)|^2 dt \leq \frac{4}{3}\frac{1}{\log R} \quad \text{for } R > 3,$$

and thus

(4.1)
$$\int_{\mathbb{R}^2} |\nabla f_R|^2 \to 0, \quad R \to \infty.$$

This completes the proof.

$\square$

The phenomenon in the above example can be explained by the result of Cheng-Yau [3, Section 1]. In [26, Definition 0.3], the authors defined the following

Definition 4.1. *A complete manifold is said to be parabolic, if it does not admit a positive Green's function. Otherwise it is said to be nonparabolic.*

Remark 4.1. According to this definition, $\mathbb{R}^n$ is a parabolic manifold if and only if $n = 2$. In particular, (4.1) follows from Proposition 4.1 below, which is a result given in [26].

Proposition 4.1. *Let Σ be a parabolic manifold. Let $B(r)$ be the ball of radius r in Σ with respect to a reference point x_0. Let $R > r > 1$. Consider the following Dirichlet problem*

$$\begin{cases} \Delta f = 0, & \text{on } B(R)\backslash B(r) \\ f = 0, & \text{on } \Sigma\backslash B(R) \\ f = 1, & \text{on } B(r). \end{cases}$$

Then we have

$$\lim_{R \to \infty} \int_{\Sigma} |\nabla f|^2 = 0.$$

$\square$

Remark 4.2. The functions f serve as the high dimensional generalization of the MacDonald functions in the paper [6, page 21]. These functions will play an important role in the next section.

The following geometric criterion of parabolicity was proved by Grigor'yan [17, 18] and Varopoulos [31] independently (cf. [24, Equation (3.1)]):

Theorem 4.1. *Let $V(t)$ be the volume of the geodesic ball $B(t)$. If Σ is nonparabolic, then*

$$\int_1^{\infty} \frac{t dt}{V(t)} < \infty.$$

In particular, if $V(t)$ is at most of quadratic growth, then Σ is parabolic.

$\square$

Corollary 4.1. *Let Σ be a smooth surface whose Gauss curvature $K \in L^1(\Sigma)$. Then Σ is a parabolic manifold of dimension 2.*

Proof. We wish to compare the volume growth rate of the geodesic ball $V(t)$ with t. To do so, first we assume that Σ has a pole and we use the polar coordinate system given by the exponential map centered at the pole to write

$$V(t) = \int_0^t \int_0^{2\pi} f(r, \theta) \, dr d\theta,$$

where under the polar coordinates, the expression of the metric becomes $ds_{\Sigma}^2 = dr^2 + f^2(r, \theta) d\theta^2$ on Σ.

It follows that

$$V'(t) = \int_0^{2\pi} f(t, \theta) \, d\theta.$$

The Jacobi equation for the exponential map gives

$$f'' + Kf = 0 \; ; \quad f(0, \theta) = 0, \; f'(0, \theta) = 1,$$

where the prime denotes derivative in the radial direction. Thus we have

$$V'''(t) = - \int_0^{2\pi} K f(t, \theta) d\theta.$$

Since K is integrable, this implies that

$$|V''(t)| \leq C$$

for some constant C. Consequently,

(4.2)
$$V(t) \leq Ct^2$$

for t large enough.

If the surface Σ doesn't have a pole, we get the similar estimate outside the cut locus with respect to a fixed reference point. Since the measure of the cut locus is zero, we get the same estimate (4.2). This is an observation of Gromov.

Thus the volume of Σ is at most of quadratic growth and it must be parabolic by Theorem 4.1.

$\square$

5. The upper bound estimate of σ_0.

The idea to estimate σ_0, the infimum of the spectrum of the Laplacian from above is to construct test functions which would provide the strict upper bound κ_1^2 (where $\kappa_1 = \pi/2a$). We may construct test functions which are continuous everywhere on Ω and smooth everywhere on Ω except on a set of measure 0. Such functions must be in $H^{1,2}(\Omega)$, which serve our purpose.

We define the quadratic form

$$(5.1) \qquad Q(\xi, \xi) = \int_\Omega |\nabla \xi|^2 d\Omega - \kappa_1^2 \int_\Omega \xi^2 d\Omega,$$

for $\xi \in H^{1,2}(\Omega)$. By the nature of the metric on Ω, we define

$$(5.2) \qquad Q_1(\xi, \xi) = \int_\Omega |\nabla' \xi|^2 d\Omega,$$

where

$$(5.3) \qquad |\nabla' \xi|^2 = \sum_{i,j=1}^n G^{ij} \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_j},$$

and

$$(5.4) \qquad Q_2(\xi, \xi) = \int_\Omega \left(\frac{\partial \xi}{\partial u}\right)^2 d\Omega - \kappa_1^2 \int_\Omega \xi^2 d\Omega,$$

where $(x_1, \cdots, x_n, u)$ are the standard coordinates in Definition 2.1. It is clear that

$$Q(\xi, \xi) = Q_1(\xi, \xi) + Q_2(\xi, \xi).$$

The test functions we shall construct will essentially be the product of a vertical function (depending only on u) and a horizontal one (depending only on $x \in \Sigma$). Let $\varphi = \psi\chi$ be a test function, where $\psi \in C_0^\infty(\Sigma)$ and χ is a smooth function of u such that $\chi(\pm a) = 0$.

Note that

$$\nabla(\chi\psi) = \chi \nabla \psi + \psi \nabla \chi.$$

By (2.1), we have $< \nabla\psi, \nabla\chi >= 0$. Thus we have

$$(5.5) \qquad \int_\Omega |\nabla (\chi\psi)|^2 = \int_\Omega \chi^2 |\nabla \psi|^2 + \int_\Omega \psi^2 |\nabla \chi|^2.$$

We wish to prove, with the suitable choice of ψ and χ, that

$$(5.6) \qquad Q(\varphi, \varphi) = \int_\Omega \varphi_u^2 - \kappa_1^2 \int_\Omega \varphi^2 + \int_\Omega \chi^2 |\nabla\psi|^2 < 0,$$

where φ_u denotes $\frac{\partial\varphi}{\partial u}$ and $\varphi_i = \frac{\partial\varphi}{\partial x_i}$ for $i = 1, \cdots, n$.

Like in the paper [6], we choose $\chi = \cos\kappa_1 u$. We need the following elementary lemma:

Lemma 5.1. *Let $a > 0$ be a positive number and let $\kappa_1 = \frac{\pi}{2a}$. Let $\chi(u) = \cos\kappa_1 u$, let*

$$(5.7) \qquad \mu_k = \int_{-a}^a u^k(\chi_u^2 - \kappa_1^2\chi^2)\, du, \forall k \geq 0.$$

Then

$$(5.8) \qquad \mu_k = \begin{cases} 0 & \text{if } k \text{ is odd, or } k = 0; \\ \frac{1}{2}\frac{(k)!}{(2\kappa_1)^{k-1}} \sum_{l=1}^{k/2} \frac{(-1)^{k/2-l}\pi^{2l-1}}{(2l-1)!}, & \text{if } k \neq 0 \text{ is even.} \end{cases}$$

Furthermore, $\mu_k > 0$ if $k \neq 0$ is even.

$\square$

Theorem 5.1. *We assume that the hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic satisfying Assumptions A1), A2). Moreover, we assume that $\sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A)$ is integrable and*

$$(5.9) \qquad \int_\Sigma \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A) d\Sigma \leq 0,$$

where A is the second fundamental form of Σ, μ_k for $k \geq 1$ is defined in Lemma 5.1, $[n/2]$ is the integer part of $n/2$, and $c_k(A)$ is the k-th elementary symmetric polynomial of A. If Σ is not totally geodesic, then

$$\sigma_0 < \kappa_1^2.$$

Proof. We first consider the test functions of the form $\varphi = \psi \cdot \chi$. We define ψ as follows: Let x_0 be a fixed point of Σ and let $R > r > 1$. Let $B(R)$ and $B(r)$ be two balls in Σ of radius R and r centered at x_0 respectively. We define ψ as

$$(5.10) \qquad \begin{cases} \Delta \psi = 0 & \text{on } B(R) - B(r); \\ \psi|_{B(r)} \equiv 1; \\ \psi|_{\Sigma - B(R)} \equiv 0, \end{cases}$$

and we define $\chi = \cos \kappa_1 u$.

By the definition of the functions χ and ψ, using Lemma 2.1, we know that there is a constant C such that

$$(5.11) \qquad \int_\Omega \chi^2 |\bigtriangledown \psi|^2 d\Omega \leq C \int_\Sigma |\nabla_\Sigma \psi|^2 d\Sigma,$$

where ∇_Σ is the connection of Σ. We first assume that

$$(5.12) \qquad \int_\Sigma \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A) d\Sigma = -\delta < 0.$$

By (2.4) and Lemma 5.1, we have

$$(5.13) \qquad \int_\Omega \psi^2 |\chi_u|^2 - \kappa_1^2 \int_\Omega \psi^2 \chi^2 = \int_\Sigma \psi^2 \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A) \, d\Sigma.$$

By the maximum principle and the fact that $\psi|_{B(r)} \equiv 1$, we have

$$(5.14) \qquad \int_\Sigma \psi^2 \sum_{k \geq 1} \mu_{2k} c_{2k}(A) \leq \int_{B(r)} \sum_{k \geq 1} \mu_{2k} c_{2k}(A) + \int_{\Sigma \backslash B(r)} |\sum_{k \geq 1} \mu_{2k} c_{2k}(A)|.$$

On the other side, since $\sum \mu_{2k} c_{2k}(A)$ is integrable, if r is large enough, by the above inequality, we have

$$(5.15) \qquad \int_\Sigma \psi^2 \sum \mu_{2k} c_{2k}(A) d\Sigma < -\frac{\delta}{2}.$$

By Proposition 4.1 and (5.11), if R large enough, we have

$$(5.16) \qquad \int_\Omega \chi^2 |\nabla \psi|^2 \leq \frac{\delta}{4}.$$

Combining (5.13), (5.15) and (5.16), we proved (5.6) under the assumption (5.12).

Now we assume that

$$(5.17) \qquad \int_\Sigma \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A) d\Sigma = 0.$$

In this case, the test functions $\varphi = \psi\chi$ are not good enough to give the upper bound of σ_0. We shall use a trick in [6] (see also [4, 29]) to construct the test functions.

We let

$$\varphi_\varepsilon = \varphi + \varepsilon j\chi_1,$$

where ε is a small number, j is a smooth function on Σ whose support is contained is $B(r-1)$, and χ_1 is a smooth function on $[-a, a]$ such that $\chi_1(\pm a) = 0$. As a general fact, we have

(5.18) $$Q(\varphi_\varepsilon, \varphi_\varepsilon) = Q(\varphi, \varphi) + 2\varepsilon Q(\varphi, j\chi_1) + \varepsilon^2 Q(j\chi_1, j\chi_1).$$

By (5.6), (5.11), and (5.13), we have

(5.19) $$Q(\varphi_\varepsilon, \varphi_\varepsilon) \le C \int_\Sigma |\nabla\psi|^2 d\Sigma + \int_\Sigma \psi^2 \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A)\, d\Sigma + 2\varepsilon Q(\varphi, j\chi_1) + \varepsilon^2 Q(j\chi_1, j\chi_1).$$

Since $\operatorname{supp} j \subset B(r-1)$, we have

(5.20) $$Q(\varphi, j\chi_1) = \int_\Omega j(\chi_u(\chi_1)_u - \kappa_1^2 \chi\chi_1)d\Omega = \int_\Sigma j \int_{-a}^a (\chi_u(\chi_1)_u - \kappa_1^2 \chi\chi_1)\det(1 - uA)dud\Sigma.$$

Using integration by parts, we have

(5.21) $$Q(\varphi, j\chi_1) = -\int_\Sigma j \int_{-a}^a \chi_u \frac{\partial}{\partial u} \det(1 - uA)\chi_1 dud\Sigma.$$

Now we are able to choose suitable j and χ_1 for our purpose. By assumption, we know that Σ is not totally geodesic. Thus at least there is a point $x \in \Sigma$ such that $\partial_u \det(1 - uA) \not\equiv 0$. We assume that $x \in B(r-1)$ without losing generality. We choose χ_1 and j such that the integral $Q(\varphi, j\chi_1)$ is not zero. Note that the choice of j is independent of φ. We choose ε (positive or negative) small enough so that

$$2\varepsilon Q(\varphi, j\chi_1) + \varepsilon^2 Q(j\chi_1, j\chi_1) < 0.$$

Finally, since

$$\operatorname{supp} j \subset B(r-1),$$

the above expression is independent to r and R. By the parabolicity of Σ, if $r, R \to \infty$, then

$$\int_\Sigma |\nabla\psi|^2 d\Sigma \to 0,$$

and by the assumption (5.17),

$$\int_\Sigma \psi^2 \sum_{k=1}^{[n/2]} \mu_{2k} c_{2k}(A)\, d\Sigma \to 0.$$

Thus by (5.19), $Q(\varphi_\varepsilon, \varphi_\varepsilon)$ is negative for r, R large. This completes the proof of the theorem.

□

Let $\mathcal{R} = (R_{ijkl})$ be the curvature tensor of Σ. Define

(5.22) $$\operatorname{tr}(\mathcal{R}^p) = \sum_{i_s < j_s, k_s < l_s, s = 1, \cdots, p} (-1)^{sgn(\sigma)} R_{i_1 j_1 k_1 l_1} \cdots R_{i_p j_p k_p l_p},$$

where σ is the permutation $(i_1, \cdots, j_p; k_1, \cdots, l_p)$. Then from Gray [15, (4.15)], we have

Proposition 5.1. *Using the above notations, we have*

$$\operatorname{Tr}(\mathcal{R}^p) = c_{2p}(A)$$

□

Remark 5.1. If n is even, then up to a constant, $\operatorname{Tr}(\mathcal{R}^{n/2}) = c_n(A)$ is the Gauss-Bonnet-Chern density.

Proof of Theorem 1.3. By Theorem 3.1, Theorem 5.1, and Proposition 5.1, we have

$$\sigma_0 < \kappa_1^2 \leq \sigma_{\text{ess}}.$$

Thus the ground state exists.

$\square$

Proof of Theorem 1.2. By the Theorem of Hartman, we know that (1.4) is equivalent to

$$\int_\Sigma K \leq 0.$$

Thus the result follows from Theorem 1.3 for $n = 2$.

$\square$

Proof of Corollary 1.1. If $n = 3$, then the conditions (1.5) are

$$\text{Tr}\,(\mathcal{R}^1) \text{ is integrable and } \int_\Sigma \text{Tr}\,(\mathcal{R}^1) \leq 0.$$

But $\rho = 2\,\text{Tr}(\mathcal{R}^1)$.

If $n = 4$, a tedious computation gives

$$\text{Tr}\,(\mathcal{R}^2) = \frac{1}{24}(\rho^2 - 4|\text{Ric}|^2 + |\mathcal{R}|^2),$$

where Ric is the Ricci curvature of Σ, and $|\text{Ric}|, |\mathcal{R}|$ are the norms of the Ricci tensor and the curvature tensor, respectively. If the sectional curvature is positive outside a compact set, then by [16, Theorem 9], $\text{Tr}\,(\mathcal{R}^2)$ is integrable and

$$\int_\Sigma \text{Tr}\,(\mathcal{R}^2) \leq \frac{4\pi^2}{3}e(\Sigma),$$

where $e(\Sigma)$ is the Euler characteristic number of Σ. The theorem follows from the above inequality, Proposition 5.1, and Theorem 1.3.

$\square$

Before finishing this section, we give the following example of the manifold Σ of dimension 3 satisfying the conditions in Theorem 1.3. Thus the theorem is not an empty statement for high dimensions.

Example 2. *There is a complete manifold Σ of dimension 3 immersed in $\mathbb{R}^4$ such that*

(1) *It is parabolic;*
(2) *$A \to 0$, where A is the second fundamental form;*
(3) *$\frac{1}{2}\int_\Sigma |\rho| = \int_\Sigma |c_2(A)| < +\infty$;*
(4) *$\frac{1}{2}\int_\Sigma \rho = \int_\Sigma c_2(A) < 0$.*

Proof. Let $\Sigma = S^1 \times \mathbb{R}^2$. We consider the immersion by

$$\Sigma \to \mathbb{R}^4, \quad (\theta, t, \varphi) \to (\sigma(t)\cos\theta, \sigma(t)\sin\theta, t\cos\varphi, t\sin\varphi),$$

where $\sigma(t)$ is a smooth positive function defined in (5.24). Here we use θ as the local coordinate of S^1 and $(x, y) \in \mathbb{R}^2$ with $x = t\cos\varphi, y = t\sin\varphi$. The Riemannian metric of Σ is

$$ds^2 = (1 + \sigma'(t)^2)(dt)^2 + \sigma^2(t)(d\theta)^2 + t^2(d\varphi)^2.$$

We claim that Σ is parabolic. In order to prove this, we let $x_0 = (1, 0, 0) \in \Sigma$. Let $B(R)$ be the geodesic ball of radius R centered at x_0. Then $B(R) \subset \{x \in \Sigma | t < R\}$. To see this, let $x \in B(R)$ such that dist $(x, x_0) = R'$, and let $\eta = (\eta_1(s), \eta_2(s), \eta_3(s))$ be the geodesic line of Σ connecting x_0 and x, where s is the arc length. Then we have

$$R = R' \geq \int_0^{R'} (1 + \sigma'(s)^2)^{\frac{1}{2}}|\eta_1'(s)|ds \geq t.$$

From the above equation, we have

$$\operatorname{vol} B(R) \le 4\pi^2 \int_0^R t\sigma(t)(1 + \sigma'(t)^2)^{\frac{1}{2}} dt \le CR^2 \log R$$

for some constant C. Thus we have

$$\int_0^\infty \frac{t}{\operatorname{vol} B(t)} dt = +\infty,$$

and Σ is parabolic by Theorem 4.1.

The normal vector of Σ in $\mathbb{R}^4$ is

$$N = \frac{1}{\sqrt{1 + \sigma'(t)^2}}(\cos\theta, \sin\theta, -\sigma'\cos\varphi, -\sigma'\sin\varphi).$$

The principal curvatures are

$$\frac{\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}}, \quad -\frac{1}{\sigma\sqrt{1 + \sigma'(t)^2}}, \quad \frac{\sigma'}{t\sqrt{1 + \sigma'(t)^2}}.$$

By the definition of $\sigma(t)$, all principal curvatures go to zero as $t \to \infty$. Thus $A \to 0$ at infinity. On the other hand

$$(5.23) \qquad \int_\Sigma c_2(A) = 4\pi^2 \int_0^\infty \left(\frac{\sigma\sigma'\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} - \frac{t\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} - \frac{\sigma'}{\sqrt{1 + \sigma'(t)^2}} \right) dt$$

We let the function $\sigma(t)$ be a smooth increasing function such that

$$(5.24) \qquad \begin{cases} \sigma(t) = \log t & t > 3 + \varepsilon \\ \sigma(t) = \log 3 & t < 3 \end{cases},$$

for ε small. The last two terms of (5.23) can be calculated easily:

$$(5.25) \qquad \int_0^\infty \left(-\frac{t\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} - \frac{\sigma'}{\sqrt{1 + \sigma'(t)^2}} \right) dt = -\frac{t\sigma'}{\sqrt{1 + \sigma'(t)^2}} \bigg|_0^\infty = -1.$$

Let R be a big number. We have

$$\int_0^R \frac{\sigma\sigma'\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} dt = -\frac{\log R}{(1 + \frac{1}{R^2})^{\frac{1}{2}}} + \log 3 + \int_{3+\varepsilon}^R \frac{\sigma'(t)}{(1 + \sigma'(t)^2)^{\frac{1}{2}}} dt + \int_3^{3+\varepsilon} \frac{\sigma'(t)}{(1 + \sigma'(t)^2)^{\frac{1}{2}}} dt.$$

The last term can be estimated by

$$\int_3^{3+\varepsilon} \frac{\sigma'(t)}{(1 + \sigma'(t)^2)^{\frac{1}{2}}} dt \le \log(3 + \varepsilon) - \log 3.$$

Thus a straightforward computation gives

$$\int_0^R \frac{\sigma\sigma'\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} dt \le -\frac{\log R}{(1 + \frac{1}{R^2})^{\frac{1}{2}}} + \log(3 + \varepsilon) + \log(R + \sqrt{1 + R^2}) - \log(3 + \varepsilon + \sqrt{1 + (3 + \varepsilon)^2}).$$

We let $R \to \infty, \varepsilon \to 0$. Then we have

$$(5.26) \qquad \int_0^\infty \frac{\sigma\sigma'\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} dt \le \log 6 - \log(3 + \sqrt{10}) < 0.$$

By (5.25) and (5.26), we have $\int_M c_2(A) < 0$. Finally, since

$$\frac{\sigma\sigma'\sigma''}{(1 + \sigma'(t)^2)^{\frac{3}{2}}} \sim O(\frac{\log t}{t^3}),$$

and

$$-\frac{t\sigma''}{(1+\sigma'(t)^2)^{\frac{3}{2}}} - \frac{\sigma'}{\sqrt{1+\sigma'(t)^2}} \sim O(\frac{1}{t^3}),$$

we know that $c_2(A)$ is integrable.

$\square$

6. CONVEX SURFACES

In this section, we consider the layer over a convex surface Σ in $\mathbb{R}^3$. Σ is defined by

$$(6.1) \qquad\qquad z = f(x,y),$$

where $f(x,y)$ is a smooth convex function, $f(0) = 0$, $\nabla f(0) = 0$, and $\nabla^2 f(0) > 0$.

The main result of this section is the following:

Theorem 6.1. *Let Σ be defined as above. Suppose $\Omega = \Sigma \times (-a, a)$ is the layer with depth $a > 0$. Then the infimum of the spectrum σ_0 satisfies*

$$(6.2) \qquad\qquad \sigma_0 < \kappa_1^2,$$

where $\kappa_1 = \pi/(2a)$.

We begin with the following

Lemma 6.1. *With the assumptions on f, there is a number $\delta > 0$ such that*

$$f_r = \frac{\partial f}{\partial r} > \delta$$

for $x^2 + y^2 \geq 1$, where (r, θ) is the polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$.

Proof. By the assumptions, there is a $\delta_0 \in (0, 1)$ such that $f_r|_{x^2+y^2=\delta_0} > 0$. By convexity, we have $f_{rr} \geq 0$. Then since the circle $\{x^2 + y^2 = \delta_0\}$ is compact, we can conclude that $f_r \geq \delta$ for $x^2 + y^2 \geq 1$.

$\square$

Corollary 6.1. *Using the above notations, we have*
   (1) $|\nabla f| \geq \delta$;
   (2) $f(x,y) \geq \delta \cdot (\sqrt{x^2 + y^2} - 1)$.

$\square$

An interesting consequence of the above corollary is the following: Let b be a large positive number. Let C_b be the curve defined by the intersection of Σ with respect to the plane $z = b$. Clearly C_b is a convex curve. From the above corollary, C_b is contained in a disk of radius $b/\delta + 1$. In particular, we have the estimate of the length of the curve

$$(6.3) \qquad\qquad \int_{C_b} 1 \leq Cb$$

for a constant C.

For the manifold Σ, the mean curvature H can be represented by

$$(6.4) \qquad\qquad H = \frac{(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy}}{(1 + |\nabla f|^2)^{3/2}},$$

where $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, and $|\nabla f|^2 = f_x^2 + f_y^2$. We compare the mean curvature to the curvature of the convex curve $f(x,y) = b$, which is given by

$$(6.5) \qquad\qquad k_b = \frac{f_{xx}f_y^2 - 2f_{xy}f_x f_y + f_{yy}f_x^2}{|\nabla f|^3}.$$

By Corollary 6.1 and the convexity of f, we have

$$(6.6) \qquad\qquad\qquad\qquad\qquad H \geq \frac{1}{2}\delta^3 k_b,$$

if δ is small enough. Since $C_b : \{f(x,y) = b\}$ is a convex curve, we have

$$(6.7) \qquad\qquad\qquad\qquad\qquad \int_{f(x,y)=b} k_b = 2\pi.$$

Thus by (6.6)

$$(6.8) \qquad\qquad\qquad\qquad\qquad \int_{f(x,y)=b} H \geq \pi\delta^3.$$

By the co-area formula (cf. [30, page 89]), we have

$$(6.9) \qquad\qquad\qquad \int_{x^2+y^2\geq 1} H d\Sigma \geq \int_c^\infty \left(\int_{f=t} \frac{H}{|\tilde{\nabla} f|}\right)dt,$$

where c is a positive real number, and

$$(6.10) \qquad\qquad\qquad\qquad\qquad |\tilde{\nabla} f|^2 = \frac{|\nabla f|^2}{1+|\nabla f|^2}$$

is the gradient of f on the *Riemannian manifold* Σ. Thus by (6.8), and Corollary 6.1,

$$(6.11) \qquad\qquad\qquad\qquad\qquad \int_{x^2+y^2\geq 1} H = +\infty.$$

Proof of Theorem 6.1. We shall again use the trick introduced by [6] (see also [4, 29]) to perturb the "standard" test functions. However, our choices of perturbation functions are quite different from theirs in nature.

Let K be the Gauss curvature of Σ. Then we have $K \geq 0$, and

$$(6.12) \qquad\qquad\qquad\qquad\qquad \int_\Sigma K \leq 2\pi$$

by the theorem of Huber [20]. Since the Gauss curvature is nonnegative, the volume growth is at most quadratic. Thus Σ is parabolic. For any $r_1 > 0$, we can find a function φ such that

(1) $\varphi \in C_0^\infty(\Sigma)$, $0 \leq \varphi \leq 1$;
(2) $\varphi \equiv 1$ on $B(r_1)$, where $B(r_1)$ is the geodesic ball of radius r_1 of Σ centered at 0;
(3) $\int_\Sigma |\nabla\varphi|^2 d\Sigma < 1$.

The quadratic forms Q, Q_1, and Q_2 are defined in (5.1), (5.2), and (5.4). Let $\chi = \cos\kappa_1 u$. Then we have

$$(6.13) \qquad\qquad Q_1(\varphi\chi, \varphi\chi) = \int_\Omega |\nabla\varphi|^2\chi^2 d\Omega \leq a(1+C_0)^2,$$

where $C_0 < 1$ is defined in Assumption A1). We also have

$$(6.14) \qquad\qquad\qquad Q_2(\varphi\chi, \varphi\chi) \leq \mu_2 \int_\Sigma K\varphi^2 d\Sigma.$$

Combining the above two equations and using (6.12), we have

$$(6.15) \qquad Q(\varphi\chi, \varphi\chi) = Q_1(\varphi\chi, \varphi\chi) + Q_2(\varphi\chi, \varphi\chi) \leq C_1,$$

where C_1 is a constant depending only on Σ and a.

Suppose r_1 is large enough such that $\{f(x,y) \leq 2R^2\} \subset B(r_1)$ for some large number $R > 0$. We consider a function $\rho(t)$ on $\mathbb{R}$ such that

(1) $\rho \equiv 1$, if $t \in [R, R^2]$;

(2) $\rho \equiv 0$ if $t > R^2 + 1$ or $t < R - 1$;

(3) $0 \leq \rho \leq 1$;

(4) $|\rho'| \leq 4$.

We define $\psi(x, y) = \rho(f(x, y))$. ψ is a smooth function of Σ. Let χ_1 be an odd function of u such that $\chi_1(\pm a) = 0$, and

(6.16)
$$\int_{-a}^{a} \chi_u(\chi_1) du = -\sigma < 0,$$

where $\sigma > 0$ is a positive number. We consider the function $\varphi\chi + \varepsilon\psi\chi_1/f$, where ε is a small number to be determined. By the definition of $Q(\cdot, \cdot)$ and (6.15), we have

(6.17)
$$Q(\varphi\chi + \varepsilon\psi\chi_1/f, \varphi\chi + \varepsilon\psi\chi_1/f) \leq C_1 + 2\varepsilon Q(\varphi\chi, \psi\chi_1/f) + \varepsilon^2 Q(\psi\chi_1/f, \psi\chi_1/f).$$

If r_1 and R are big, then

$$\operatorname{supp}\psi \subset \{x \in \Sigma \mid \varphi(x) \equiv 1\}.$$

We thus have

$$\int_{\Omega} \langle \nabla'(\varphi\chi), \nabla'(\psi\chi_1/f) \rangle d\Omega = 0,$$

where ∇' is defined in (5.3). By (2.4), we have

$$d\Omega = (1 - Hu + Ku^2) d\Sigma.$$

Since χ_1 is odd and χ is even, by the above equation, we have

$$Q(\varphi\chi, \psi\chi_1/f) = \int_{\Omega} (\chi_u(\chi_1)_u \varphi\psi/f - \kappa_1^2 \chi\chi_1 \varphi\psi/f) d\Omega = -\int_{-a}^{a} u(\chi_u(\chi_1)_u - \kappa_1^2 \chi\chi_1) du \int_{\Sigma} \psi H/f d\Sigma.$$

Since

$$\int_{-a}^{a} u(\chi_u(\chi_1)_u - \kappa_1^2 \chi\chi_1) du = -\int_{-a}^{a} \chi_u(\chi_1) du,$$

we have

(6.18)
$$Q(\varphi\chi, \psi\chi_1/f) = -\sigma \int_{\Sigma} \frac{\psi H}{f} d\Sigma,$$

where σ is the number defined in (6.16). By the co-area formula and (6.8), (6.10), we have

(6.19)
$$\int_{\Sigma} \frac{\psi H}{f} = \int_{\mathbb{R}} \frac{\rho(t)}{t} \left(\int_{f=t} \frac{H}{|\tilde{\nabla}f|} \right) dt \geq \pi\delta^3 \int_{\mathbb{R}} \frac{\rho(t)}{t} dt \geq \pi\delta^3 \log R.$$

In order to estimate the last term of (6.17), we first note that

(6.20)
$$\left| \tilde{\nabla} \frac{\psi}{f} \right| \leq C_2/f$$

for some constant C_2, where $\tilde{\nabla}$ is the covariant derivative of Σ. From this, we have

(6.21)
$$Q_1\left(\frac{\psi\chi_1}{f}, \frac{\psi\chi_1}{f} \right) \leq C_3 \int_{R-1 \leq f \leq R^2+1} 1/f^2 d\Sigma$$

for some constant C_3. Using the same argument, we have

(6.22)
$$Q_2\left(\frac{\psi\chi}{f}, \frac{\psi\chi}{f} \right) \leq C_4 \int_{R-1 \leq f \leq R^2+1} 1/f^2 d\Sigma$$

for some constant C_4. We use the co-area formula again to estimate

(6.23)
$$\int_{R-1 \leq f \leq R^2+1} 1/f^2 = \int_{R-1}^{R^2+1} \frac{1}{t^2} \left(\int_{f=t} \frac{1}{|\tilde{\nabla}f|} \right) dt.$$

From Corollary 6.1, we know that $|\tilde{\nabla}f|$ has a lower bound. Thus by (6.3), there is a constant C_5 such that

$$\tag{6.24} \int_{R-1 \leq f \leq R^2+1} 1/f^2 d\Sigma \leq C_5 \log R.$$

Thus by (6.18), (6.19), (6.21), (6.22), and (6.24), from (6.17), we have

$$\tag{6.25} Q(\varphi\chi + \varepsilon\psi\chi_1/f, \varphi\chi + \varepsilon\psi\chi_1/f) \leq C_1 - 2\varepsilon\pi\sigma\delta^3 \log R + \varepsilon^2 C_5(C_3 + C_4)\log R.$$

We choose ε to be a small positive number such that

$$-2\varepsilon\pi\sigma\delta^3 + \varepsilon^2 C_5(C_3 + C_4) < 0.$$

We then let R large enough (which requires r_1 be large enough also). Then the left hand side of (6.25) is negative. By the definition of σ_0, we know that $\sigma_0 < \kappa_1^2$.

$\square$

Proof of Theorem 1.1. By Theorem 3.1 and Theorem 6.1, we have

$$\sigma_0 < \kappa_1^2 \leq \sigma_{\text{ess}}.$$

Thus the ground state exists.

$\square$

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